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13. Abstract *(Maximum 200 Words)* (abstract should contain no proprietary or confidential information)

Research was performed on the p- and hp-versions of the finite element method with focus on reliable determination of margins of safety for structural components of aircraft. The investigation focused on structural failure by loss of stability and time dependent effects that can be modeled with linear viscoelastic constitutive laws. The main results of the project were: (a) The formulation of the problem of stability in a fully three-dimensional setting; implementation in a research code, and application to problems of substantial complexity. (b) The formulation and analysis of small-strain viscoelasticity in the context of the mixed p-version of the finite element method. It was shown that p-extensions provide a basis for the implementation of efficient and reliable quality assurance procedures.

A summary of the project is presented in this document with minimal technical details. The technical details can be found in the referenced publications.

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RESEARCH ON THE p- AND hp-VERSIONS OF THE FINITE ELEMENT METHOD

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Introduction

This project was concerned with the development of mathematical methods and computational procedures for reliable quantitative determination of margins of safety (MS) for aircraft structural systems. By definition:

$$MS = \frac{\phi_{\mathrm{ALL}}}{\phi_{\mathrm{MAX}}} - 1 \tag{1}$$

where $\phi_{\mathrm{ALL}} > 0$ is the allowable value of a functional prescribed by design specifications and $\phi_{\mathrm{MAX}} > 0$ is the value of that functional for a given loading condition. The principal objective of this project was to develop procedures for the estimation and control the errors associated with numerical determination of ϕ_{MAX} and proper interpretation of experiments aimed at the definition of ϕ_{ALL}. The investigation focused on two areas: Failure through loss of stability in composite plates and shells and failure of adhesively bonded joints in composite materials subjected to thermal and mechanical loads. There are many important practical applications, such as electronic packaging, ceramics, adhesively bonded joints and laminated composites.

The specific objectives of the project were:

1. Investigate p-adaptive procedures. The goal was to identify specific regions on the solution domain where a trial discretization needs to be modified.
2. Develop model-adaptive procedures for structural plates and shells made of laminated composites.

3. Develop methods and procedures for the numerical determination of the eigenpairs that characterize the generalized stress intensity factors along re-entrant edges in heterogeneous solid bodies.

4. Develop a method for numerical determination of the generalized stress intensity factors in heterogeneous bodies subjected to thermal and mechanical loading.

5. Develop procedures for the simulation of time-dependent material response (creep and relaxation).

One of the difficulties in creating reliable design criteria for structures made of composite materials is that composite materials can fail by a variety of mechanisms. At present it is not possible to predict with a reasonable degree of certainty and consistency when and how a structure made of composite materials will fail. Failure may be caused by delamination, separation of the plies, buckling of the fibers, failure in the adhesive, loss of structural stability, or some combination of these mechanisms. Owing to the complexity of the problem, estimation of margins of safety by numerical means is not within the state of the art at present. Progress is possible through carefully designed physical experiments coupled with numerical solution of the models that represent the experiments. The numerical solutions must have a guaranteed accuracy. The development of reliable methods of analysis was a central consideration in this work. The main points are described in the following section.

Allowables

The definition and determination of allowable values ϕ_{ALL} in eq. (1) requires close correlation between experimental and computed data. The procedure involves the formulation of a hypothesis that failure occurs when a certain functional ϕ reaches a critical value. The functional ϕ cannot be observed directly, therefore it must be inferred from correlations between computed values and experimental observations.

Let Y_{ij} be the ith ideal observation of the jth experiment and let $\phi_i(u_{EX}^{(j)})$ be the corresponding functional computed from the exact solution $u_{EX}^{(j)}$ so that if there were no experimental errors and the mathematical model and the hypothesis were correct then we would have

$$Y_{ij} - \phi_i(u_{EX}^{(j)}) = 0.$$

Due to experimental errors e_{ij} and inherent uncertainties we actually observe $y_{ij} = Y_{ij} \pm e_{ij}$ and compare this with $\phi_i(u_{FE}^{(j)})$ where $u_{FE}^{(j)}$ is the finite element solution corresponding to the jth experiment. Writing:

$$Y_{ij} - \phi_i(u_{EX}^{(j)}) = \underbrace{Y_{ij} \pm e_{ij}}_{y_{ij}} \mp e_{ij} - \phi_i(u_{EX}^{(j)}) + \phi_i(u_{FE}^{(j)}) - \phi_i(u_{FE}^{(j)})$$

and using the triangle inequality, we have

$$\underbrace{|Y_{ij} - \phi_i(u_{EX}^{(j)})|}_{\text{true error}} \leq \underbrace{|y_{ij} - \phi_i(u_{FE}^{(j)})|}_{\text{apparent error}} + |e_{ij}| + \underbrace{|\phi_i(u_{FE}^{(j)}) - \phi_i(u_{EX}^{(j)})|}_{\text{error of discretization}}. \tag{2}$$

This result shows that to test a particular hypothesis, it is essential to have both the experimental errors $|e_{ij}|$ the discretization errors $|\phi_i(u_{FE}^{(j)}) - \phi_i(u_{EX}^{(j)})|$ under control, otherwise it will not be possible to know whether the apparent error is due to an error in the hypothesis being tested; errors in the numerical approximation, or errors in the experiment. Furthermore, control of discretization errors, in terms of the data of interest, must be intrinsic to the numerical solution method. The discretization errors must be much less than the experimental errors.

Inherent uncertainties in the data may be a dominant factor. In that case the aim of the experiments should include the development of reliable statistical information.

Summary of accomplishments

The results of this research have been published, or are being processed for publication. A summary of the work completed under this project is presented in the following.

1. **p-Adaptive methods.** In the application of the p-version of the finite element method the finite element mesh is fixed and the polynomial degree of the elements is increased until a desired level of precision is reached. Ideally, the mesh should be designed taking into account *a priori* information concerning the smoothness of the solution. Uniform meshes are optimal where the solution is smooth. In the neighborhood of singularities geometrically graded elements are optimal and in dimensionally reduced problems, such as plates and shells, the mesh design should account for boundary layer effects as well. In the case of large problems it is not feasible in general to create meshes that account for singularities everywhere. The goal is to identify critical regions, called the regions of primary interest, where stresses or stress intensity factors have to be computed. In many cases the regions of primary interest are known *a priori*. In the other parts of the domain only a reasonably accurate representation of the compliances is required. These parts are called the regions of secondary interest. Compliances are well approximated when the error in strain energy is small.

   The goal of adaptivity is to ensure that the error in energy norm is small and the error is nearly equidistributed among the finite elements.

   Many finite element analysis codes, including the three largest commercial codes, now offer p-extension capabilities. In most industrial applications p-extension is used in conjunction with mesh generators that typically generate nearly uniform meshes of tetrahedral elements. A capability for adaptive p-distribution is needed to ensure nearly optimal performance for a given number of degrees of freedom.

Existing *a posteriori* error estimators can be grouped into three large categories: estimators based on (a) residuals, (b) stress smoothing, and (c) extrapolation. Estimators of type (a) and (b) are used in conjunction with the h- and hp-versions, whereas estimators of type (c) require information generated by p-extensions.

Construction of local error indicators based on element-residuals as well as stress smoothing and patch recovery have been the focus of intensive research. Indicators based on residuals typically require local equilibration of the residuals and jumps in tractions at inter-element boundaries. The error is then estimated by solving a local (i.e., element-by-element) problem using either the principle of minimum potential energy, or its dual, the principle of minimum complementary energy, on a space which is larger than the one used for obtaining the finite element solution. The space on which the minimum is sought can be enlarged by p- or h-extension. The difficulty with achieving local equilibrium is that the jump discontinuities must be carefully distributed such that all jumps are accounted for and each element is in equilibrium under the action of the distributed load corresponding to the residuums and the assigned tractions corresponding to the jump discontinuities in stresses along the perimeter.

One important advantage of the method developed in this project is that equilibration is not required. Recognizing that in the complementary energy principle the displacements are the natural boundary conditions, which can be easily applied, the local problem is formulated using the displacements computed from the finite element solution and a stress space which satisfies both the equilibrium and compatibility conditions. The complementary potential energy is then minimized on this space. The method can be applied on individual elements, a group of elements, or part of an element. The error is measured by the work of the jump in tractions at element boundaries computed from the solution of the dual problem, using the displacement computed from the finite element solution (the primal) and the strain energy of the difference in stresses computed from the primal and the dual on the element domain. The sum of the error measures assigned to the element edges and to the elements is used as global error estimator. The ratio of this global error measure and the total potential energy is found to be close to the square of the relative error in energy norm for a wide range of problems. Details and examples are available in [2].

2. **Model-adaptive methods for laminated plates and shells.** An investigation of model adaptive procedures for laminated plates and shells was undertaken. The main points are outlined in a highly simplified setting. Details are available in [1].

   Consider an elastic strip of rectangular cross-section and assume that the x-y plane is a principal plane of the cross section and the beams are subjected to bending and shear in the x-y plane only and the axial force is zero. The material is assumed to be elastic and isotropic. Considering this as a two-dimensional problem of elasticity, we

can write the displacement vector components u_x and u_y in the form:

$$u_x = u_{x|1}(x)\phi_1(y) + u_{x|3}(x)\phi_3(y) + u_{x|5}(x)\phi_5(y) + \cdots + u_{x|(2m-1)}(x)\phi_{2m-1}(y) \quad (3)$$

$$u_y = u_{y|0}(x)\psi_0(y) + u_{y|2}(x)\psi_2(y) + u_{y|4}(x)\psi_3(y) + \cdots + u_{y|(2n-2)}(x)\psi_{2n-2}(y) \quad (4)$$

where ϕ_i (resp. ψ_i) are antisymmetric (resp. symmetric) director functions chosen so that an optimal rate of convergence is realized in the sense of eq. (7). The functions $u_{x|k}(x)$, $u_{y|k}(x)$ are called field functions.

Writing the displacement components in this form allows one to consider a family of hierarchic models (semi-discretizations) for beams and laminated elastic strips characterized by m and n. The highest member of the hierarchy is the problem of two-dimensional elasticity which corresponds to $m = n = \infty$. Denoting the exact solution of the hierarchic model characterized by m and n by $\vec{u}_{EX}^{(m,n)}$ and the exact solution of the problem of two-dimensional elasticity by $\vec{u}_{EX}^{(2D)}$, the hierarchic models satisfy the following three criteria:

(a) Approximability:

$$\lim_{\substack{m\to\infty \\ n\to\infty}} ||\vec{u}_{EX}^{(2D)} - \vec{u}_{EX}^{(m,n)}||_E = 0. \quad (5)$$

where $||\cdot||_E$ is the energy norm.

(b) Asymptotic consistency:

$$\lim_{h\to 0} \frac{||\vec{u}_{EX}^{(2D)} - \vec{u}_{EX}^{(m,n)}||_E}{||\vec{u}_{EX}^{(2D)}||_E} = 0. \quad (6)$$

(c) Optimality of convergence:

$$\frac{||\vec{u}_{EX}^{(2D)} - \vec{u}_{EX}^{(m,n)}||_E}{||\vec{u}_{EX}^{(2D)}||_E} \approx Ch^{\gamma(m,n)} \quad \text{as } h \to 0,\ m,n \to \infty \quad (7)$$

with convergence rates $\gamma(m+1,n) \geq \gamma(m,n)$; $\gamma(m,n+1) \geq \gamma(m,n)$.

The formulation and analysis of hierarchic models for plates and shells was completed. This is documented in [1].

3. **Asymptotic expansions.** The extraction of the coefficients of the eigenpairs (the generalized flux intensity factors or GFIFs and the generalized stress intensity factors or GSIFs) from the finite element solution, at multi-material interfaces, as shown schematically in Fig. 1, based on the L_2 and energy projection method on a circular sector, was investigated [15]. The problem of coupled thermo-elastic problems, accounting for the dependence of material properties on temperature was investigated

and implemented for problems of heat conduction. To solve this nonlinear problem, in the first step a linear heat conduction problem is solved using an average value for the coefficients of thermal expansion. In subsequent steps the thermal conductivity tensor is determined from the temperature of the preceding iteration step at each integration point and, using these values, the stiffness matrix is re-computed. The stopping criterion is based on the difference of two consecutive solution vectors [6], [15].

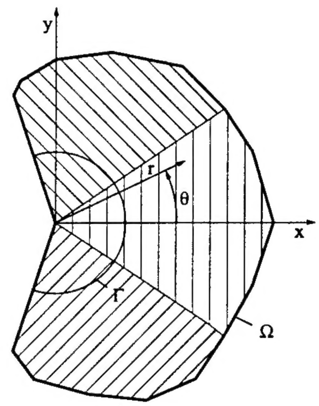

Figure 1: Typical multi-material interface.

4. **Nonlinear models: Average stress/strain criteria.** Real materials typically have a periodic internal structure characterized by the grain size, molecular length, fiber diameters and the spacing of the fibers. The theory of elasticity is not applicable on length scales smaller than a few periods, called the *characteristic length*. In composite materials it is often necessary to account for material nonlinearities, however, especially at adhesively bonded joints. For these reasons the use of the average stress as the basis for the development of failure criteria is far more realistic than the determination of the first term of the asymptotic expansion. Nevertheless, it is very useful to have the asymptotic expansion computed, since the average stress on a small area can be computed very efficiently from the asymptotic expansion.

   In the linear case it can be seen that criteria based on stress or strain averages are equivalent to the criteria based on stress intensity factors. Specifically: the average stress $\tilde{\sigma}(\vec{n}, \Delta A_n)$ is defined as:

$$\tilde{\sigma}(m, \Delta A_n) := \frac{1}{\Delta A_n} \iint_{\Delta A} \sigma_{ij} m_i n_j \, dA$$

   where ΔA_n is a small area the orientation of which is characterized by the unit vector n_i. The direction of the stress acting on ΔA_n in given by the unit vector m_i, as shown in Fig. 2.

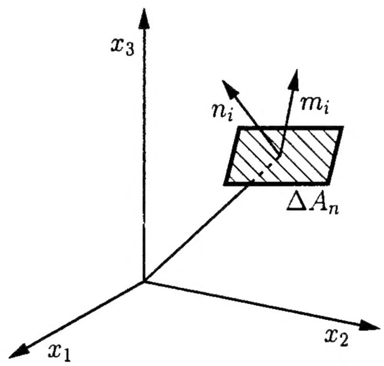

Figure 2: Notation.

In linear elastic fracture mechanics (LEFM) this is equivalent to computing the stress intensity factor. For example, considering a cracked plate of thickness t and making the usual assumption that the crack is aligned with the x_1-axis and the crack tip is located in the origin, we have:

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + O(r^{1/2}).$$

Letting $m_i = n_i = \{0\ 1\}$ (i.e., $\theta = 0$) and $\Delta A_n = t\Delta\ell$ we have

$$\tilde{\sigma} = \frac{1}{t\Delta\ell}\int_0^{\Delta\ell} \frac{K_I}{\sqrt{2\pi r}}\, t\, dr = \frac{2K_I}{\sqrt{2\pi\Delta\ell}} + O(\Delta\ell^{1/2}).$$

It is seen that the average stress on a fixed small area, characterized in this example by $\Delta\ell$, is proportional to the stress intensity factor K_I. The proportionality constant depends on $\Delta\ell$. In a perfectly homogeneous and isotropic material $\Delta\ell$ is an arbitrary small distance selected such that the $O(\Delta\ell^{1/2})$ terms are negligible in comparison with the first term. Having a capability for computing average stresses for linear as well as nonlinear models makes it possible to investigate a variety of failure criteria of the type

$$\frac{\tilde{\sigma}}{\tilde{\sigma}_{MAX}} + \frac{\tilde{\tau}}{\tilde{\tau}_{MAX}} \leq 1$$

where $\tilde{\sigma}$ (resp. $\tilde{\tau}$) is the normal (resp. shear) stress on a given small area [12], [11], [4], [5].

In addition, the problem of mechanical contact was investigated using adaptive meshing and space enrichment methods [7].

5. **Limits of elastic stability.** The increasing emphasis on affordability in military systems has led to a number of advances in airframe design and production. Unitized airframe structural components are replacing sheet metal built-up structures in order to reduce part count and assembly cycle times and costs. The qualification of thick aluminum plate stock enables the machining of integral structure with a minimum of expensive tooling. The development of high speed machining (HSM) techniques has further enabled the fabrication of thin lightweight structure to provide improved performance along with affordability. A unitized structural component is shown in Fig. 3. The web thickness is in the range 2.5 to 5 mm, the flange thickness is between 2.5 and 3 mm. The development of methods for the design and certification of unitized structural components based on numerical simulation is of obvious importance. One of the failure modes is loss of stability.

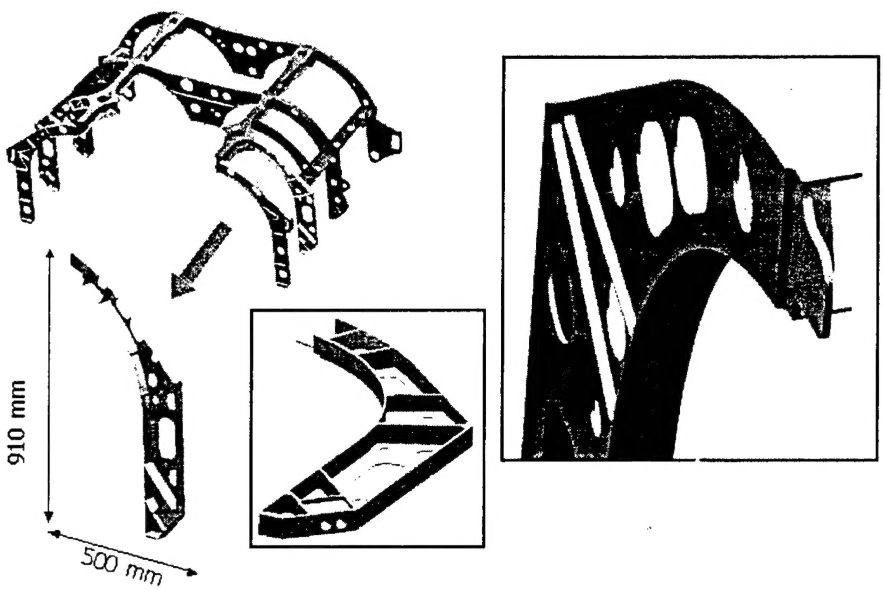

Figure 3: A unitized structural component.

In many cases failure of structural components is caused by some form of instability, such as buckling, crippling and flutter. Reliable mathematical models that account for instability will make it possible to obtain improved estimates of the margin of safety of structural components. The mathematical models used in current professional practice are based on dimensionally reduced descriptions of an elastic body. Such models cannot represent the initial stress state where the assumptions incorporated in dimensionally reduced models do not hold, and hence may lead to engineering decisions that cause unexpected failures or avoidable weight penalties.

An investigation of linearized models of structural stability was completed. The formulation of the problem, documented in reference [3], requires the computation of the characteristic values of non-compact operators. While the numerical results indicate that a broad range of important practical problems can be solved by this method, questions of a theoretical nature remained. In particular, it was not known whether spurious eigenvalues would appear when the initial stresses are bounded, and how singular points might pollute the computed eigenvalues. Of practical, engineering interest are problems where the domain, or a large part of the domain, is thin, as in the example shown in Fig. 3. This class of problems is of particular interest in aerospace applications where weight considerations dictate efficient use of material. In a collaborative work with Professor Manil Suri of the University of Maryland, Baltimore County, these problems were substantially resolved.

Two mathematical models of the general theory of elastic stability exist in the classical literature: These models are called the Biot-Prager model and the Trefftz model. Both were implemented in fully three-dimensional setting so that numerical experiments could performed. For thin structures the results closely matched the classical results. It was found that the two classical models yield virtually identical results.

Some fundamental questions concerning the existence of a solution, the properties of the spectrum, and their relationship to loss of stability had not been investigated. In this investigation two working hypotheses were advanced [3]:

(a) The spectrum is a point spectrum, hence it is meaningful to consider the lowest nonzero eigenvalue as an indicator of the onset of instability;

(b) The minimal eigenvalue of the finite dimensional problem converges to its infinite dimensional counterpart as the finite element space is enlarged (i.e., the degrees of freedom are increased).

In a parallel investigation the both working hypothesis were proven by Professors Manil Suri, Ivo Babuška and Monique Dauge. The second hypothesis was shown to hold subject to the condition that the stress field is bounded and the solution domain is thin [14]. In many practical problems this condition cannot be satisfied, because inclusion of fillets, needed for eliminating stress singularities in the elastic solution

6. **Viscoelastic models.** Investigation of the problem of viscoelastic solids was undertaken. The goal is to represent the time-dependent behavior of materials using the p- and hp-versions of the finite element method and establish a robust, reliable method for concurrent control of the spatial and temporal errors. Since the constitutive equations are of the form

$$\mathbf{P}(D)\sigma = \mathbf{Q}(D)\epsilon$$

where $\mathbf{P}(D)$ and $\mathbf{Q}(D)$ are differential operators in time and $\boldsymbol{\sigma}$ (resp. $\boldsymbol{\epsilon}$) is the stress (resp. strain) tensor, the mixed method must be used. The mixed formulation in the context of the p-version has been implemented for planar and axisymmetric prolems in an experimental program and model problems have been solved.

An important result, obtained recently by C. Schwab and his associates[1] is that for diffusion problems exponential rate of convergence can be realized when the discontinuous Galerkin method (DG) is employed utilizing the hp-version in space and time.

In viscoelastic problems and problems of plasticity the stresses are not computable from the displacement field. For this reason it is necessary to employ mixed formulations, that is, formulations in which the displacement and stress fields are approximated separately.

First we consider the problem of linear isotropic elasticity. The solution domain is denoted by Ω. The body force vector is denoted by f_i and the tractions acting on the boundary Γ is denoted by g_i. The displacement field is denoted by $\mathbf{u} \equiv u_i$, $(i = 1, 2)$. The strain tensor $\epsilon_{ij}^{\mathbf{u}}$ is defined by:

$$\epsilon_{ij}^{\mathbf{u}} \overset{\text{def}}{=} \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{8}$$

The stress tensor is denoted by $\boldsymbol{\sigma} \equiv \sigma_{ij}$ and the formulation is stated as follows: Find $(\mathbf{u}, \boldsymbol{\sigma}) \in H^1(\Omega) \times H^0(\Omega)$ such that for all $(\mathbf{v}, \mathbf{w}) \in H^1(\Omega) \times H^0(\Omega)$,

$$\iint_\Omega \sigma_{ij}\epsilon_{ij}^{\mathbf{v}} t dS = \iint_\Omega f_i v_i t dS + \int_\Gamma g_i v_i t dL \tag{9}$$

$$\iint_\Omega \epsilon_{ij}^{\mathbf{u}} w_{ij} t dS = \iint_\Omega \sigma_{ij} A_{ijkl} w_{kl} t dS \tag{10}$$

where $H^k(\Omega)$ represents the usual Sobolev space with k generalized derivatives on Ω, with $L_2(\Omega) = H^0(\Omega)$. The norm of $H^k(\Omega)$ will be denoted by $\|\cdot\|_{H^k}$. A_{ijkl} is a symmetric positive definite tensor[2] which depends on the Young's modulus E and Poisson's ratio ν and whether planes stress or plane strain conditions are assumed, t is the thickness.

In the numerical solution we use the spaces $V_h = V_h(\Delta, p) \subset H^1(\Omega)$, $S_h = S_h(\Delta, q) \subset H^0(\Omega)$ where Δ is the finite element mesh and p (resp. q) is the polynomial degree

[1] Schwab, C. and Schötzau, D., "Time Discretization of Parabolic Problems by the hp-Version of the Discontinuous Galerkin Finite Element Method", Research Report No. 99-04 Seminar für Angewandte Mathematik, ETH, Zürich, 1999.

[2] $A_{ijkl}\sigma_{ij}\sigma_{kl} > 0$ for any $\sigma_{ij} \neq 0$ in $H^0(\Omega)$.

assigned to the elements for displacements (resp. stresses). Introducing the notation:

$$b(\boldsymbol{\sigma}, \mathbf{v}) \stackrel{\text{def}}{=} \iint_{\Omega} \sigma_{ij} \epsilon_{ij}^{\mathbf{v}} t dS$$
$$a(\boldsymbol{\sigma}, \mathbf{w}) \stackrel{\text{def}}{=} \iint_{\Omega} \sigma_{ij} A_{ijkl} w_{kl} t dS$$
$$G(\mathbf{v}) \stackrel{\text{def}}{=} \iint_{\Omega} f_i v_i t dS + \int_{\Gamma} g_i v_i t dL$$

the $(\mathbf{u}, \boldsymbol{\sigma})$ formulation can be written in the following form: Find $(\mathbf{u}, \boldsymbol{\sigma}) \in V_h \times S_h \in H^1 \times H^0$ such that

$$a(\boldsymbol{\sigma}, \mathbf{w}) - b(\mathbf{w}, \mathbf{u}) = 0 \qquad \forall\ \mathbf{w} \in S_h \tag{11}$$
$$b(\boldsymbol{\sigma}, \mathbf{v}) = G(\mathbf{v}) \quad \forall\ \mathbf{v} \in V_h. \tag{12}$$

In order to establish that the system of equations corresponding to (11) and (12) is solvable, it is necessary to show that the Babuška-Brezzi condition, also known as the inf-sup condition, is satisfied:

$$\sup_{\boldsymbol{\sigma} \in \mathcal{N}_h} \frac{a(\boldsymbol{\sigma}, \mathbf{w})}{\|\boldsymbol{\sigma}\|_{H^0}} \geq C \|\mathbf{w}\|_{H^0} \quad \forall\ \mathbf{w} \in \mathcal{N}_h \subset H^0(\Omega) \tag{13}$$

$$\sup_{\boldsymbol{\sigma} \in S_h} \frac{b(\boldsymbol{\sigma}, \mathbf{v})}{\|\boldsymbol{\sigma}\|_{H^0}} \geq C \|\mathbf{v}\|_{H^1} \quad \forall\ \mathbf{v} \in V_h \subset H^1(\Omega) \tag{14}$$

where the space $\mathcal{N}_h \subset S_h$ is defined by:

$$\mathcal{N}_h \stackrel{\text{def}}{=} \{\boldsymbol{\sigma} \in S_h \text{ such that } b(\boldsymbol{\sigma}, \mathbf{v}) = 0\ \forall\ \mathbf{v} \in V_h\}.$$

Noting that $a(\boldsymbol{\sigma}, \mathbf{w})$ is coercive on $H^0 \times H^0$, inequality (13) is satisfied. Inequality (14) is clearly satisfied if we can select σ_{ij} proportional to $\epsilon_{ij}^{\mathbf{v}}$. For quadrilateral elements, product space, and triangular elements this is possible when $q \geq p-1$. For quadrilateral elements, trunk space[3], this is possible if $q \geq p$, unless the trunk space is so constructed that the shape functions span all polynomials of degree q or less, but not more than degree q. In other words, the $x^q y$ and xy^q terms are omitted, in which case $q \geq p-1$. Omission of the terms $x^q y$ and xy^q is possible because continuity conditions are not enforced on S_h.

Remark 1 Strictly speaking, the rules for solvability apply to linear mapping only. In the p version, where generally large elements are used, curved elements are necessary

[3] Also known as 'serendipity' space.

to represent curved domains accurately. The numerical examples presented in this paper indicate that the algebraic equations are solvable when smooth mapping is used. This is consistent with the results obtained by Chilton and Suri[4] where it is shown that mixed curved quadrilateral elements retain the robustness of their straight-sided counterparts, provided that degenerate elements are avoided.

Summary

The problem of establishing criteria for the computation of margins of safety for laminated composites and adhesively bonded joints requires (a) the construction of a mathematical model that accounts for the essential physical attributes; (b) reliable numerical solution of the mathematical model, and (c) experimental determination of the allowable values. At present the principles underlying the construction of mathematical models are well understood. Efficient methods for computing the data of interest to within a prescribed tolerance are now available commercially and research is being performed at various academic institutions aimed at improving the efficiency of numerical solution methods. A great deal of work remains to be done in the development of experimental data for the allowable values, however. The experimental work must be coupled with reliable numerical solution methods, the foundations for which have been laid in this research project.

Personnel supported

Faculty:	Dr. Barna A. Szabó	Professor of Mechanics (PI)
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	Dr. Roland Krause	part time
	Dr. George Kiralyfàlvi	part time
Graduate students:	Ms. Li Zhang	
	Mr. Konrad Juethner	
	Mr. Dominik Scholz	

Interactions/Transitions

Industrial application has been made possible by Engineering Software Research and Development, Inc. (ESRD), located in St. Louis, Missouri. ESRD, founded in 1989 by Barna Szabó, Ivo Babuška and Kent Myers develops, documents and maintains an advanced professional quality finite element analysis code called *StressCheck* which is the principal channel by which the results of AFOSR-sponsored research performed at the Center for

[4] Chilton L, Suri M. On the construction of stable curvilinear p version elements for mixed formulations of elasticity and Stokes' flow. *Numer. Math.* 2000; **86**:29–48.

Computational Mechanics of Washington University have been transitioned into professional practice.

The current users of StressCheck include The Boeing Aircraft Company (several divisions); Northrop Grumman Corporation; Lockheed Martin Corporation; Cessna Aircraft Co.; New Piper Aircraft Co., Raytheon, NASA Johnson Space Center, Israel Aircraft Industries, and others.

Specific examples of recent applications by persons not connected to this project are:

1. Report by C. L. Brooks et al., entitled "Corrosion is a Structural and Economic Problem: Transforming Metrics to a Life Prediction Method". In this report it is shown how StressCheck is used for the prediction of life of corroded aircraft components. The work described in the report was performed under the sponsorship of the U.S. Air Force.

2. Paper by Arnold Nathan of the Israel Aircraft Industries: "Composite Repair of Aging Metallic Structure: p-Version 3D Finite Element Approach", presented at the 2nd Joint NASA/FAA/DoD Conference on Aging Aircraft, 31 August-3 September, 1998 Williamsburg, VA. This paper describes a typical application of StressCheck to the analysis composite material repair patch bonded to a metallic structure.

3. Paper S. P. Engelstad and R. L. Actis [13] The desire to reduce costs of composite aircraft structure has recently increased the importance of bonded joint analysis capabilities in the aeronautics industry. A new initiative has been formed, called the Composites Affordability Initiative (CAI), with the goal to decrease the recurring acquisition costs of composite airframe structure by 50%, thereby making composites more affordable for the next generation of fighter aircraft. Participants in the initiative include the Air Force, Navy, Lockheed Martin, Boeing, and Northrop Grumman. New validated analysis tools for composite bonded joints have been viewed as the key enabler for the application of innovative manufacturing and bonding technologies for airframe primary structure. Thus, improved analysis tools for design of bonded composite joints have become the focus of a CAI analysis tools team. This paper reviews the development history and describes new capabilities of a p-version Finite Element-based tool developed by a joint effort of the CAI team and a software provider.

   The capability of performing rapid parametric sizing studies was one of the central goals of the team. It was desired to develop parameterized solutions for various bonded joint configurations, and validate failure predictions with CAI test data. These solutions would then be used to perform rapid, preliminary design sizing of composite bonded joints. Bonded joint analysis tools previously existed in the aeronautics industry (such as the industry standard A4EI code), but are limited to simple joint geometry (typically single, double, and step-lap joints) due to the one dimensional nature of the

solution. Much more complex joints are being utilized to achieve affordability goals, thus solution techniques require at a minimum two-dimensional continuum elasticity analysis. Many finite element studies utilizing plane strain analysis appear in the literature, all utilizing h-element finite element models. Many utilized nonlinear models of the adhesive, and others included geometric nonlinearity. Different methods for handling the so-called spew fillet geometry, which is formed at the adhesive free edge, have also been considered. Uncertainty of the shape of this region has been studied also. Since singularities exist in the elasticity solution at free edges of composite layer interfaces, and also at reentrant corners, consideration of the convergence issues and error control in these regions is important. It is the authors' opinion that more attention to error control in these regions is required, since failure tends to initiate in the vicinity of these singularities. Other important issues include the need for a modern, user-friendly interface, abilities for the user to build-in failure criteria, advanced stress/strain extraction capabilities, and insensitivity to very large aspect ratios caused by the modeling of thin composite layers.

The CAI analysis tools team completed an extensive study of the state-of-the-art in the area of composite stress analysis tools, as would be applied to failure analysis of composite bonded joints. The software StressCheck, developed by Engineering Software Research and Development, Inc. (ESRD), was identified as a potential provider, and following an evaluation phase, was selected. This software offers a p- and hp-element formulation, which satisfies the need for error control, aspect ratio insensitivity, and also includes geometric and material nonlinearity. StressCheck already had an electronic handbook interface, which contained many of the required parameterization features. Once StressCheck became an approved CAI analysis tool for further development, CAI and ESRD worked closely to define StressCheck modifications required for the bonded joint applications. ESRD has implemented CAI requested software improvements through both industry contract funding and AFOSR sponsorship[5]. This paper presents a review of the modifications implemented into StressCheck, and a model problem consisting of a double-lap joint analysis. The differences between the new and conventional modeling procedures, including the handling of singularities, and error control, are identified.

This paper also illustrates an example of a very fruitful effort of the CAI team in identifying and filling an analysis tool requirement. An important conclusion is the need for industry users to work closely with software providers to precisely tailor software to the requirements of the problem and the focus user group. This experience highlights the key link being forged between aerospace industry researchers and software providers to develop next generation analysis tools.

[5]Project No. FQ8671-9501469 STTR/TS.

The Principal Investigator presented the results obtained under this project to defense contractors and an Air Force Laboratory:

1. The Boeing Company, St. Louis, MO (contact person: Mark Holly 314-232-1405) August 18, 1998.

2. Materials and Manufacturing Directorate, AFRL WPAFB July 21, 1998 (contact persons: Dr. Richard B. Hall 937-255-9097, Dr. Nicholas Pagano 937-255-1138)

3. Lockheed Martin, Marietta, GA. Met with the Composites Affordability Initiative (CAI) Group CAI is a government/industry consortium. Members represent each of the major US aerospace companies and the Air Force and Navy. The principal investigator made a presentation to a committee of CAI charged with the responsibility of developing procedures for the reliable and efficient numerical simulation of structural components fabricated of composite materials. Following the presentation an in-depth study of the numerical procedures developed under AFOSR sponsorship at the Center for Computational Mechanics of Washington University was undertaken by CAI. (Contact person: Dr. Steven P. Engelstad 770-494-9714) March 3, 1998. Frequent-follow-ups.

4. Presentation to The Boeing Company St. Louis: "Engineering decisions based on computed information: Time, quality and cost" April 7, 2000 (contact person: Eric Meyer 314-232-4097)

5. Boeing Aircraft Company, Seattle, WA "New Directions, Methods and Tools for Stress Analysis" 17 Oct. 2000

6. Boeing Aerospace, Long Beach, CA "Quality Assurance in Engineering Computations. Applications to structural design and analysis", October 16, 2000 (contact person: Patrick Goggin 563-593-7367).

The Principal Investigator delivered lectures/presentations at the following professional meetings:

- Szabó, B. and Actis, R. L., "Hierarchic Models in the Numerical Simulation of Mechanical Systems", Keynote Lecture, 4th World Congress on Computational Mechanics, Buenos Aires, Argentina, 29 June - 2 July 1998.

- Szabó, B. A. and Actis, R. L, "Analysis of bonded and fastened repairs by the p-version of the finite element method", The Fourth Joint DoD/FAA/NASA Conference on Aging Aircraft, St. Louis, MO May 15-18, 2000.

- Seminars at three German Universities: Munich Stuttgart and Hannover at the invitation of Professor Erwin Stein. Seminar titles; "Numerical Simulation of Mechanical Systems; Problems of Reliability and Complexity" and "Model-Adaptive Processes in the Numerical Simulation of Mechanical Systems", March 7-12, 2000.

- Invited lecture: "Quality Assurance in the Numerical Simulation of Mechanical Systems", Mathematische Analyse von FEM für Probleme in der Mechanik, Mathematisches Forschungsinstitut, Oberwolfach, Germany, February 1999.

- Invited paper: "Mathematical Models of Fastened Structural Connections" SAE General, Corporate and Regional Aviation Meeting, Wichita, KS, 20-22 April 1999

- Invited minisymposium presentation: "Linear models for estimating the limits of elastic stability" 1999 SIAM Annual Meeting, Atlanta, GA May 1999.

- Invited plenary lecture: "Standardization of Design and Analysis Procedures" Implementation Road Map Conference 1999, Dearborn, Michigan, October 1999.

- Keynote address: "Hierarchic modeling: Concepts and implementation", Advanced Design and Analysis Conference, Tokyo, Japan December 16, 2000.

- Invited paper: "Quality Assurance in the Numerical Simulation of Mechanical Systems", The 5th International Conference on Computational Structures Technology, Leuven, Belgium, September 2000.

- Contributed paper: "Analysis of Bonded and Fastened Joints by the p-Version of the Finite Element Method" The Fourth Joint DoD/FAA/NASA Conference on Aging Aircraft, St. Louis 15-18 May, 2000

- Invited presentation: "New FEA-Based Technology for Integral Structures", Aircraft Structural Integrity Program Conference, Integral Structure Technical Exchange, San Antonio, Texas, December 4, 2000

New discoveries, inventions and patent disclosures

The findings of this research are essential for quantitative assessment of the strength of homogeneous and heterogeneous solid bodies. Failure theories require experimental correlation and confirmation before they could be properly called discoveries, however.

Honors/Awards

The principal investigator received an honorary doctorate (doctor honoris causa, 1998). He is an external member of the Hungarian Academy of Sciences and Fellow of the US

Association for Computational Mechanics. The principal investigator serves on the editorial boards of Computers and Structures, and Engineering with Computers, and on the scientific advisory boards of the First M.I.T. Conference on Computational Fluid and Solid Mechanics (June 2001).

Publications

[1] Actis, R. L., Szabó, B. A. and Schwab, C., "Hierarchic Models for Laminated Plates and Shells", *Comput. Methods Appl. Mech. Engrg.*, Vol. 172, pp. 79-107 (1999)

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[4] Szabó, B. and Actis, R., "Hierarchic Models in the Numerical Simulation of Mechanical Systems", Proc. 4th World Congress on Computational Mechanics, Buenos Aires, Argentina, 29 June - 2 July 1998

[5] Szabó, B. and Actis, R., "Mathematical Models of Fastened Structural Connections", Paper No. 1999-01-1576, Society of Automotive Engineers, Wichita, KS, 1999.

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[9] Szabó, B., Actis, R. and Babuška, I., "Comments on MIL-HDBK-17-3E, Working Draft", Technical Note, Center for Computational Mechanics, Washington University, June 2000.

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[11] Actis, R. and Szabó, B., "Mathematical models of fastened structural connections" First Canadian Conference on Nonlinear Solid Mechanics Victoria, BC June 16-20, 1999

[12] Actis, R. and Szabó, B., "Analysis of bonded and fastened repairs by the p-version of the finite element method", to appear in *Computers and Mathematics with Applications*, 2001.

[13] Engelstad, S. P. and Actis, R. L., "Development of p-version handbook solutions for analysis of composite bonded joints", to appear in *Computers and Mathematics with Applications*, 2001.

[14] Suri, M. Private communications.

[15] Yosibash, Z., Actis, R. and Szabó, B., "Extracting edge flux intensity functions for the Laplacian", to appear in *Int. J. Numer. Meth. Engng.*